

Absolutely trianalytic tori in the generalized Kummer variety

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Abstract

We prove that a generic complex deformation of a generalized Kummer variety contains no complex analytic tori.

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1 Introduction

A Riemannian manifold is called *hyperkähler* if it admits a triple of complex structures I, J, K satisfying quaternionic relations and Kähler with respect to g .

Definition 1.1: A manifold M is called *holomorphically symplectic* if it is a complex manifold with a closed holomorphic 2-form Ω over M such that $\Omega^n = \Omega \wedge \Omega \wedge \dots \wedge \Omega$ is a nowhere degenerate section of a canonical class of M , where $2n = \dim_{\mathbb{C}}(M)$.

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A hyperkähler manifold is always holomorphically symplectic. By the Yau's Theorem [Y], a hyperkähler structure exists on a compact complex manifold if and only if it is Kähler and holomorphically symplectic.

Given any triple $a, b, c \in \mathbb{R}$, $a^2 + b^2 + c^2 = 1$, the operator $L := aI + bJ + cK$ satisfies $L^2 = -1$ and defines a Kähler structure on (M, g) . Such a complex structure is called *induced by the hyperkähler structure*. Complex subvarieties of such (M, L) for generic (a, b, c) were studied in [V1], [V2].

Definition 1.2: A compact hyperkähler manifold M is called *simple*, or *irreducible holomorphically symplectic (IHS)* or *of maximal holonomy* if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Theorem 1.3: (Bogomolov's decomposition, [B]) Any hyperkähler manifold admits a finite covering which is a product of a torus and several hyperkähler manifolds of maximal holonomy.

Definition 1.4: Let M be a hyperkähler manifold, and all induced complex structures $L := aI + bJ + cK$, where $a, b, c \in \mathbb{R}$, $a^2 + b^2 + c^2 = 1$ fit together into a family over \mathbb{CP}^1 called *the twistor family* of complex structures.

Definition 1.5: A closed subset Z of a hyperkähler manifold M is called *trianalytic* if it is complex analytic with respect to complex structures I, J, K .

Definition 1.6: Let (M, I, J, K) be a compact, holomorphically symplectic, Kähler manifold, and $Z \subset (M, I)$ a complex subvariety which is trianalytic with respect to any hyperkähler structure compatible with I . Then Z is called *absolutely trianalytic*.

Whenever L is a generic element of a twistor family, all subvarieties of (M, L) are trianalytic (see Theorem 3.7).

Absolutely trianalytic subvarieties were studied in [V3], where it was shown that a general deformation of a Hilbert scheme of a K3 surface has no complex (or, equivalently, no absolutely trianalytic ([V3, Theorem 8.5.])) subvarieties. However, there is an absolutely trianalytic subvariety in a generalized Kummer variety. Recently, Soldatenkov and Verbitsky [SV] have shown that there are no absolutely trianalytic tori in the 6- and 10-dimensional O'Grady examples. Non-existence of absolutely trianalytic subvarieties of known type in 10-dimensional O'Grady manifold M follows from [SV, Corollary 3.17]. Non-existence of absolutely trianalytic tori in a

6-dimensional O'Grady manifold follows from representation theory of Clifford algebras [SV]. Automorphisms of hyperkähler manifolds acting trivially on the second cohomology group can be characterized as those which have absolutely trianalytic graph in $M \times M$ for any given hyperkähler structure on M . The group of such automorphisms is finite [H1]. It has been studied for the generalized Kummer surfaces by Oguiso ([Og]), Boissiere, Nieper-Wisskirchen and Sarti ([BNS]), and for O'Grady examples by Mongardi and Wandel ([MW]).

In the present paper we study absolutely trianalytic tori in the generalized Kummer variety. In Section 2 we recall all known examples of hyperkähler manifolds. In Section 3 we study general trianalytic subvarieties. In Section 4 we show non-existence of absolutely trianalytic tori in the generalized Kummer variety and prove the following

Theorem 1.7: Let $K_n(T)$ be a generalized Kummer variety, and $Z \subset K_n(T)$ be an absolutely trianalytic subvariety of $K_n(T)$. Then Z is not a torus.

2 Preliminaries

Let (M, I, J, K) be a hyperkähler manifold, and let $\omega_I, \omega_J, \omega_K$ be the corresponding Kähler forms.

A simple algebraic calculation [Bes] shows that the following form

$$\Omega = \omega_J + \sqrt{-1}\omega_K \quad (2.1)$$

is of type $(2, 0)$. Since it is closed this form is also holomorphic and moreover nowhere degenerate, as another linear algebraic argument shows. It is called *the canonical holomorphic symplectic form of a manifold M* . Thus, the underlying complex manifold (M, L) is holomorphically symplectic for each hyperkähler manifold M and an induced complex structure L . The converse is also true:

Theorem 2.1: ([Bea], [Bes, Chapter 11]) Let M be a compact holomorphically symplectic Kähler manifold with the holomorphic symplectic form Ω , a Kähler class $[\omega] \in H^{1,1}(M)$ and a complex structure I . Let $n = \dim_{\mathbb{C}} M$. Assume that $\int_M \omega^n = \int_M (\operatorname{Re} \Omega)^n$. Then there is a unique hyperkähler structure $(I, J, K, (\cdot, \cdot))$ over M such that the cohomology class of

the symplectic form $\omega_I = (\cdot, I\cdot)$ is equal to $[\omega]$ and the canonical symplectic form $\omega_J + \sqrt{-1}\omega_K$ is equal to Ω .

Two-dimensional irreducible holomorphic symplectic manifolds are $K3$ surfaces. In higher dimensions there are only few examples known. Here is the list of known examples, where compact manifolds of the same deformation type are not distinguished.

(0) *K3 surface.*

- (i) *The Hilbert scheme of n points of $K3$.* If X is a $K3$ surface then the Hilbert scheme $\text{Hilb}^n(X)$ is an irreducible holomorphic symplectic manifold [Bea]. Its dimension is $2n$ and for $n > 1$ its second Betti number is equal to 23. Details of construction of the Hilbert scheme can be found, for example [Bea]. Namely, let X be a $K3$ surface. Take the symmetric product $X^{(r)} = X^r/\mathfrak{S}_r$ which parametrizes subsets of r points in a $K3$ surface X , counted with multiplicities; it is smooth on the open subset X_0 consisting of subsets with r distinct points, but singular otherwise. We blow up singular locus and obtain a smooth compact manifold. This is the Hilbert scheme $X^{[r]}$. The natural map $X^{[r]} \rightarrow X^{(r)}$ is an isomorphism above X_0 , and it resolves the singularities of $X^{(r)}$. Alternatively, the Hilbert scheme parametrizes all 0-dimensional subschemes of the length n .

Let us describe the simplest case $\text{Hilb}^2(X)$ explicitly. For any surface X the Hilbert scheme $\text{Hilb}^2(X)$ is the blow-up $\text{Hilb}^2(X) \rightarrow S^2(X)$ of the diagonal

$$\Delta = \{\{x, x\} \mid x \in X\} \subset S^2(X) = \{\{x, y\} \mid x, y \in X\}.$$

Equivalently, $\text{Hilb}^2(X)$ is the $\mathbb{Z}/2\mathbb{Z}$ -quotient of the blow-up of the diagonal in $X \times X$. Since for a $K3$ surface there exists only one $\mathbb{Z}/2\mathbb{Z}$ -invariant two-form on $X \times X$, the holomorphic symplectic structure on $\text{Hilb}^2(X)$ is unique.

- (ii) *The generalized Kummer variety.* If T is a complex torus of dimension two, then the generalized Kummer variety $K_n(T)$ is an irreducible holomorphic symplectic manifold [Bea]. Its dimension is $2n$ and for $n > 2$ its second Betti number is 7. Note that the Hilbert scheme $T^{[n]}$ of a two-dimensional torus has the same properties as $K3^{[r]}$, but it is not simply connected. The commutative group structure on the torus

T defines a summation map

$$s(t_1, \dots, t_n) = t_1 + \dots + t_{n+1},$$

$$\Sigma : T^{n+1} \rightarrow T,$$

which induces a morphism $\Sigma : T^{[n+1]} \rightarrow T$. It is easy to see that Σ coincides with the Albanese map. The generalized Kummer variety $K_n(T)$ associated to the torus T is the preimage $\Sigma^{-1}(0) \subset T^{[n+1]}$ of the zero $0 \in T$. It is a hyperkähler manifold of dimension $2n$.

- (iii) *O'Grady's 10-dimensional example [O1]*. Let again X be a $K3$ surface, and M the moduli space of stable rank 2 vector bundles on S , with Chern classes $c_1 = 0, c_2 = 4$. It admits a natural compactification \bar{M} obtained by adding classes of semi-stable torsion free sheaves. It is singular along the boundary, but O'Grady [O1] constructs a desingularization of \bar{M} which is a new hyperkähler manifold, of dimension 10. Its second Betti number is 24 [R]. Originally, it was proved that it is at least 24 [O1].
- (iv) *O'Grady's 6-dimensional example [O2]*. A similar construction can be done starting from rank 2 bundles with $c_1 = 0, c_2 = 2$ on a 2-dimensional complex torus, this gives new hyperkähler manifold of dimension 6 as in (iii). Its second Betti number is 8.

Thus we have two series, (i) and (ii), and two sporadic examples, (iii) and (iv). All of them have different second Betti numbers. It has been proved ([KLS], Theorem B) that up to a deformation the moduli spaces for all sets of numerical parameters give $\text{Hilb}^n(K3)$, O'Grady examples, or do not admit a smooth symplectic resolution of singularities.

3 Subvarieties in hyperkähler manifolds

3.1 Trianalytic subvarieties

Definition 3.1: A closed subset Z of a hyperkähler manifold M is called *trianalytic* if it is complex analytic with respect to complex structures I, J, K .

Theorem 3.2: (equivalent to the Theorem 2.1)

Let M be a hyperkähler manifold. Then there exists a unique hyperkähler metric in a given Kähler class.

Definition 3.3: Let (M, I, J, K) be a compact, holomorphically symplectic, Kähler manifold, and $Z \subset (M, I)$ a complex subvariety which is trianalytic with respect to any hyperkähler structure compatible with I . Then Z is called *absolutely trianalytic*.

Theorem 3.4: [SV] For any hyperkähler manifolds M, M' in the same deformation class there is a diffeomorphism which sends absolutely trianalytic subvarieties to absolutely trianalytic.

Definition 3.5: A hyperkähler manifold is called *general* if all its subvarieties are absolutely trianalytic.

Remark 3.6: General deformation of a hyperkähler manifold is general in the sense of Definition 3.5 ([KV-book, Proposition 2.14]).

Theorem 3.7: ([V2]) Let M be a hyperkähler manifold, S its twistor family (see Definition 1.4). Then there exists a countable subset $S_1 \subset S$, such that for any complex structure $L \in S \setminus S_1$, all compact complex subvarieties of (M, L) are trianalytic.

3.2 Trianalytic subvarieties in the Hilbert scheme and O'Grady examples

Here we survey the known results about trianalytic and absolutely trianalytic subvarieties.

It was shown by Verbitsky that a general deformation of a Hilbert scheme of a $K3$ surface has no complex subvarieties [V3]. The same theorem was also claimed (Kaledin, Verbitsky) in the case of generalized Kummer varieties [KV]. However, later ([KV1]) they found that there are counterexamples in the latter case due to involution $\nu : t \rightarrow -t$ of a torus. This involution is extended to an involution of the Hilbert scheme $T^{[n+1]}$, and since it commutes with the Albanese map $T^{[n+1]} \rightarrow T$, the map ν preserves $K_n(T)$. Moreover, ν sends the Kähler class to itself. Hence, the involution ν preserves the hyperkähler structure on $K_n(T)$. For odd $n = 2m - 1$ the map ν fixes the $2m$ -tuple

$$(x_1, -x_1, x_2, -x_2, \dots, x_m, -x_m) \in T^{(n+1)}$$

When $x_i, -x_i$ are pairwise distinct, they give a point of the Hilbert scheme fixed by ν . Consider the closure X of the set of such points. It is one of

components of fixed point set of involution map ν . The submanifold X is birationally equivalent to the Hilbert scheme of a $K3$ surface.

Corollary 3.8: The variety $\mathrm{Sym}^2(T)$ contains a Kummer $K3$ surface.

Non-existence of absolutely trianalytic subvarieties in the Hilbert scheme $\mathrm{Hilb}(K3)$ of $K3$ was used in the book [KV-book] to prove compactness of deformation spaces of certain stable holomorphic bundles on M .

Theorem 3.9: ([V5, Theorem 6.2.]) Let M be a hyperkähler manifold, $Z \subset M$ a trianalytic subvariety, and I an induced complex structure. Consider the normalization

$$\widetilde{(Z, I)} \rightarrow (Z, I)$$

of (Z, I) . Then $\widetilde{(Z, I)}$ is smooth, and the map $\widetilde{(Z, I)} \rightarrow M$ is an immersion, inducing a hyperkähler structure on $\widetilde{(Z, I)}$.

This gives that any trianalytic subvariety $Z \rightarrow M$ has a smooth hyperkähler normalization \widetilde{Z} immersed to M ; this immersion is generically bijective onto its image. Therefore, we can replace any trianalytic cycle by an immersed hyperkähler manifold. In this paper, we will consider absolutely trianalytic varieties whose normalization is the torus.

Theorem 3.10: [SV] Let M be a hyperkähler manifold, $Z \subset M$ an absolutely trianalytic subvariety, and $\widetilde{Z} \rightarrow M$ its normalization such that $\widetilde{Z} = T \times \coprod_i K_i$, where K_i are IHS of maximal holonomy. Then $b_2(T) \geq b_2(M)$ and $b_2(K_i) \geq b_2(M)$.

Theorem 3.11: [SV] Let M be a hyperkähler manifold of maximal holonomy, T a hyperkähler torus, and $T \rightarrow M$ a hyperkähler immersion with absolutely trianalytic image. Then

$$\dim_{\mathbb{C}}(T) \geq 2^{\frac{b_2(X)-1}{2}}.$$

4 Tori in the generalized Kummer varieties

In this section we prove the following

Main Theorem (Theorem 1.7:) Let $K_n(T)$ be a generalized Kummer variety, and let $Z \subset K_n(T)$ be an absolutely trianalytic subvariety. Then Z is not a torus.

4.1 Flat tori in T^n

Let $Z \subset K_n(T)$ be an absolutely trianalytic subvariety whose normalization is a torus.

Remark 4.1: Since $K_n(T)$ is embedded in the Hilbert scheme of a torus $T^{[n]}$, we can consider Z as an absolutely trianalytic submanifold in $T^{[n]}$.

Consider the diagram:

$$\begin{array}{ccccc}
 Z & \longrightarrow & \pi(Z) & \longleftarrow & \tau^{-1}(\pi(Z)) \\
 \downarrow & \searrow & \downarrow & & \downarrow \\
 T^{[n]} & \xrightarrow{\pi} & T^{(n)} & \xleftarrow{\tau} & T^n,
 \end{array} \tag{4.1}$$

where $T^{[n]}$ is the Hilbert scheme of a torus, $T^{(n)}$ is the symmetric power of a torus, the map π is the Hilbert-Chow map, τ is the quotient map $T^n \rightarrow T^{(n)}$, and the square is Cartesian.

Remark 4.2: By [EV, Theorem 7.7] one can choose T in the same deformation class that is general in the sense of Definition 3.5. Recall that by Theorem 3.4 trianalytic subvarieties have the following property: for each M, M' in the same deformation class there exist diffeomorphism $M \rightarrow M'$ which sends absolutely trianalytic subvarieties to absolutely trianalytic ones. Therefore it is sufficient to prove our proposition for a general torus T .

Proposition 4.3: Let $Z \subset T^{[n]}$ be an absolutely trianalytic torus. Then each irreducible component of $\tau^{-1}(\pi(Z))$ is a general torus in T^n , $\text{Pic}(\tau^{-1}(\pi(Z))) = 0$, and the map $\tau : \tau^{-1}(\pi(Z)) \rightarrow \pi(Z)$ is finite.

Proof: Recall that T is a general torus. Therefore, all subvarieties in T^n are absolutely trianalytic, hence $\tau^{-1}(\pi(Z))$ is totally geodesic, and hence is flat. Therefore each irreducible component of $\tau^{-1}(\pi(Z))$ is a subtorus in T^n . Since T is general, any subtorus is general, and in particular the Picard group of $\tau^{-1}(\pi(Z))$ is zero. The finiteness of the map τ is clear. ■

Proposition 4.4: Let $Z \subset T^{[n]}$ be an absolutely trianalytic torus. Then the map $\pi : Z \rightarrow \pi(Z)$ is generically finite.

Proof: There exist a canonical stratification on every symplectic singularity and this stratification coincide with stratification by diagonals on $T^{(n)}$ ([K, Proposition 3.1.]). Moreover, strata carry symplectic forms, and for $T^{(n)}$ these forms are induced by the transposition-equivariant symplectic form on T^n . The restriction of the symplectic form to the smooth locus of the preimage $\pi^{-1}(V)$ for an arbitrary stratum V in $T^{(n)}$ is the pullback of a symplectic form on this stratum in $T^{(n)}$ ([K, Lemma 2.9.]). Then a dense open subset U in $Z \subset T^{[n]}$ is projected into the open part of some stratum. Thus, the restriction of the symplectic form to U is a pullback of the symplectic form on the stratum. If Z is projected to $\pi(Z)$ with a positive-dimension general fibres, then the form cannot be non-degenerate, so that Z is not symplectic. This gives a contradiction. ■

Proposition 4.5: Let $Z \subset T^{[n]}$ be an absolutely trianalytic torus. Consider the diagram

$$\begin{array}{ccc}
 & \tilde{Z} & \\
 \swarrow & & \searrow \\
 Z & & \tau^{-1}(\pi(Z)) \\
 \searrow & & \swarrow \\
 & \pi(Z) &
 \end{array} \tag{4.2}$$

where \tilde{Z} is the fibered product of Z and $\tau^{-1}(\pi(Z))$. Then Z and any component of $\tau^{-1}(\pi(Z))$ are isogenic tori.

Proof:

The fibered product \tilde{Z} is a subvariety in the product $Z \times \tau^{-1}(\pi(Z))$ of general tori. Then \tilde{Z} is trianalytic, and therefore flat. Maps of flat tori \tilde{Z} to Z and $\tau^{-1}(\pi(Z))$ are generically finite (finiteness of a map $\tilde{Z} \rightarrow \tau^{-1}(\pi(Z))$ follows from Proposition 4.4), hence these tori are isogenic. ■

Fix an irreducible component of $\tau^{-1}(\pi(Z))$ and denote it by Z' .

Denote the generic degree of the map $Z \rightarrow \pi(Z)$ by d and the degree of $\tau^{-1}(\pi(Z)) \rightarrow \pi(Z)$ by \tilde{d} .

Remark 4.6: It follows from Proposition 4.3 and Proposition 4.5 that

$$\text{Pic}(Z) = 0$$

4.2 Non-existence of trianalytic tori in the generalized Kummer variety

In this section we prove the Main Theorem 1.7.

Definition 4.7: The holomorphic symplectic volume of a holomorphic symplectic manifold (M, Ω) is $\text{Vol}_M^s := \int_M \Omega^{\frac{1}{2} \dim M} \wedge \overline{\Omega}^{\frac{1}{2} \dim M}$.

Definition 4.8: The Kähler volume of a Kähler manifold (M, I, ω) is $\frac{1}{2^{2n}(2n)!} \int_M \omega^{2n}$, where $\dim_{\mathbb{R}}(M) = 2n$.

Remark 4.9: For hyperkähler manifolds ([GV, Theorem 5.3.]) the Kähler volume is equal to the holomorphic symplectic one (the *hyperkähler condition*).

Theorem 4.10: Let $K_n(T)$ be a generalized Kummer variety, and let $Z \subset K_n(T)$ be an absolutely trianalytic subvariety of $K_n(T)$. Then Z is not a torus.

Proof:

First, let us remark that any complex structure of Kähler type on a flat torus T defines a complex structure of Kähler type on $T^{[n]}$. Consider the standard map from $H^2(T^{(n)}, \mathbb{C}) \oplus \mathbb{C}[E]$ to $H^2(T^{[n]}, \mathbb{C})$, where E is the exceptional divisor of the blow-up $T^{(n)}$ to $T^{[n]}$. Recall that the cohomology class $[E]$ is of type $(1, 1)$.

Fix a hyperkähler structure (I, J, K) on T , and let Ω be the corresponding holomorphic symplectic form on T^n . Hyperkähler triple on T^n is given by three forms ω'_I, ω'_J , and ω'_K of the same Kähler volume. Denote by $[\omega'_I]$, $[\omega'_J]$, and $[\omega'_K]$ their cohomology classes. Since the symplectic form on T^n is transposition-equivariant, the corresponding cohomology classes on $T^{(n)}$ denoted by $[\omega_I]$, $[\omega_J]$, and $[\omega_K]$ are such that

$$\tau^*[\omega_I] = [\omega'_I], \tau^*[\omega_J] = [\omega'_J], \tau^*[\omega_K] = [\omega'_K].$$

By Proposition 4.4 on each open stratum of $T^{(n)}$ there exist a symplectic form, and the cohomology class of this form is the restriction of $[\omega_J] + i[\omega_K]$ to the stratum.

It is well-known (see e.g. [OVV, Lemma 3.4]) that there exist a Kähler metric with Kähler class $[\omega_{T^{[n]}}] = [\pi^* \omega_{T^{(n)}}] - \epsilon[E]$, where E is the exceptional divisor and $0 < \epsilon < 1$.

Recall that the symplectic volume does not change under the blow-up. By Theorem 2.1 there exists some constant λ and a hyperkähler structure on $T^{[n]}$ such that $[\tilde{\omega}_I] := \lambda[\pi^*\omega_I] - \lambda\epsilon[E]$, $[\tilde{\omega}_J]$, and $[\tilde{\omega}_K]$ have the same Kähler volume. After blowing up the symmetric power $T^{(n)}$ to the Hilbert scheme of points $T^{[n]}$ pullbacks of $[\omega_J]$ and $[\omega_K]$ are $[\tilde{\omega}_J]$ and $[\tilde{\omega}_K]$:

$$\pi^*[\omega_J] = [\tilde{\omega}_J], \quad \pi^*[\omega_K] = [\tilde{\omega}_K].$$

Note that $\pi^*(\omega) \cup [E] = 0$, and let $\mu = [E]^{2n}$.

Then,

$$\text{Vol}_{\omega_I} = \int_{T^{[n]}} \tilde{\omega}_I^{2n} = (\lambda)^{2n} \cdot \text{Vol}_{\omega_I} - \lambda\epsilon^{2n} \cdot \mu$$

The constant λ can be determined from the equation above, and we have $\lambda > 1$.

Recall that by Proposition 4.4, and Proposition 4.3, the map from Z to $\pi(Z)$ and the map from Z' to $\pi(Z)$ are generically finite. Thus, symplectic volumes of Z and Z' differ by the multiplication by $\frac{\tilde{d}}{d}$

$$\frac{\tilde{d}}{d} \cdot \text{Vol}_Z^s = \text{Vol}_{Z'}^s,$$

where d is the degree of the map $Z \rightarrow \pi(Z)$ and \tilde{d} is the generic degree of the map $\tau^{-1}(\pi(Z)) \rightarrow \pi(Z)$. The Kähler volumes of Z and Z' are determined from the hyperkähler condition.

Since Z' is absolutely trianalytic in T^n , its volume with respect to ω'_J and ω'_K is equal to the volume with respect to ω'_I . However, Z is also absolutely trianalytic in $T^{[n]}$, hence this volume is also equal to the volume with respect to $\lambda[\pi^*\omega_I] - \lambda\epsilon[E]$.

Note that $[Z] \cup [E] = 0$. Indeed, consider line bundle $\mathcal{O}(E)$ restricted to Z . Since Z has zero Picard group (Remark 4.6), then there are no non-trivial line bundles over Z .

Hence, we have from the hyperkähler condition

$$1 = \frac{\int_Z (\lambda[\pi^*\omega_I] - \lambda\epsilon[E])^k}{\int_Z (\tilde{\omega}_J)^k} = \frac{\lambda^k \int_{\pi(Z)} [\omega_I]^k}{\int_{\pi(Z)} [\omega_J]^k} = \frac{\lambda^k \int_{Z'} (\omega'_I)^k}{\int_{Z'} (\omega'_J)^k} = \lambda^k.$$

On the other hand $\lambda > 1$, that gives a contradiction. ■

Remark 4.11: In the general case of absolutely trianalytic subvarieties of generalized Kummer manifold the proof above does not work. Indeed, generally Z is not isogenic to irreducible components of $\tau^{-1}(\pi(Z))$, and Picard group of Z could be non-zero.

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LABORATORY OF ALGEBRAIC GEOMETRY AND ITS APPLICATIONS,
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